URBAN COMPRESSION PATTERNS: FRACTALS AND NON-EUCLIDEAN GEOMETRIES — INVENTORY AND PROSPECT[1]

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Abstract. Urban growth and fractality is a topic that opens an entrance for a range of radical ideas: from the theoretical to the practical, and back again. We begin with a brief inventory of related ideas from the past, and proceed to one specific application of fractals in the non-Euclidean geometry of Manhattan space. We initialize our discussion by inventorying selected existing knowledge about fractals and urban areas, and then presenting empirical evidence about the geometry of and movement in physical urban space.

Selected empirical analyses of minimum path distances between places in urban space indicate that its metric is best described by a general Minkowskian one whose parameters are between those for Manhattan and Euclidean space. Separate analyses relate these results to the fractal dimensions of the underlying physical spaces. One principal implication is that theoretical, as well as applied, ideas based upon fractals and the Manhattan distance metric should be illuminating in a variety of contexts. These specific analyses are the focus of this paper, leading a reader through analytical approaches to fractal metrics in Manhattan geometry. Consequently, they suggest metrics for evaluating urban network densities as these represent compression of human activity. Because geodesics are not unique in Manhattan geometry, that geometry offers a better fit to human activity than do Euclidean tools with their unique geodesic activities: human activity often moves along different paths to get from one place to another.

Real-world evidence motivates our specific application, although an interested reader may find the subsequent "prospect" section of value in suggesting a variety of future research topics that are currently in progress. Does "network science" embrace tools such as these for network compression as it might link to urban function and form? Stay tuned for forthcoming work in *Geographical Analysis*.

Key words: fractal, non-Euclidean, Manhattan, urban growth, Minkowskian

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A rich and massive literature exists about fractals, fractal dimensions, and the use of fractals in the social and behavior sciences. One goal of the selective discussion in Part 1 of the present paper is to emphasize parts of this literature relevant to its theme.

1. Inventory

1.1. A brief history

Cities often serve to compress human activity: through economies of scale, shared rides, public transit, or Planned Unit Developments, mixed use developments, or any of a host of contemporary planning trends. When an urban analysis tool also compresses pattern or data, it fits the environment – and the outcome is likely to make good sense.

One early paper addressing this topic (Arlinghaus & Nystuen 1990) illustrates the importance of using fractal geometry as a compression tool to analyze site development of fragile land, maximizing desirable use while minimizing environmental damage. Another set of papers examines the use of fractal geometry to capture central place geometry, and employs that connection to answer remaining open questions associated with that geometry (e.g., Arlinghaus 1985, Arlinghaus & Arlinghaus 1989).

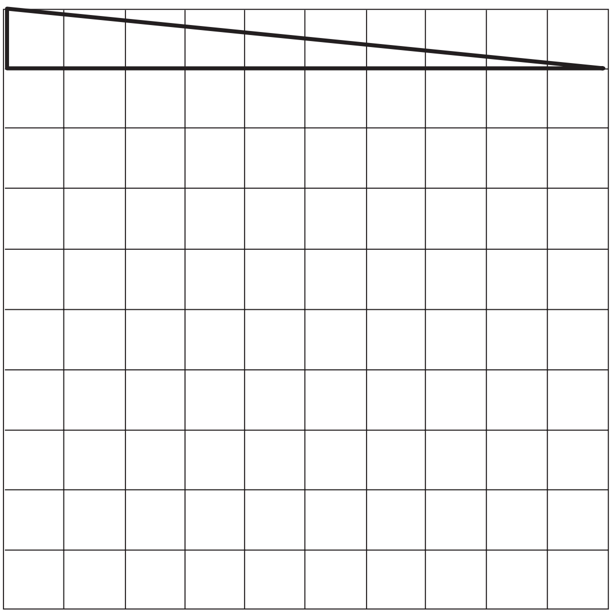

Fig. 1. Differences between Manhattan and Euclidean distances. Black line denotes a Euclidean distance.

Fractal geometry is one vehicle for capturing such compression. Another is to adjust the geometry of the underlying space containing form (Arlinghaus & Batty 2006, 2010). Maurits Escher explores the power of this approach with his 'circle limit' series. In it, the Poincaré disk model of the hyperbolic plane, which compresses all of this non-Euclidean geometry inside a single bounded compact disk, represents artistic pattern. The implications of this non-Euclidean model for compressing elements of urban form or function within a compact space serve as one way to study bounded or unbounded urban complexity within a bounded geometric environment (Arlinghaus & Nystuen 1991, Arlinghaus 2010).

However, all of these approaches function in the abstract world of pure mathematics: in it, form comes to life in its 'cleanest' pattern, without perturbations from such complications as human decisions or physical landscapes. In addition, real-world examples are critical to inform scholars of the possible utility of such approaches. Batty & Longley, and others, pursued this approach with remarkable success over a period of years. The focus here is on one particular non-Euclidean model, Manhattan space (see Fig. 1), as an intervening position between the purely abstract and the totally 'real' (Griffith et al. 2010).

1.2. An overview of fractal dimensions

The problem being addressed in this paper concerns the fractal dimension of Manhattan space, and whether or not Manhattan geometry (or some other non-Euclidean geometry) should replace Euclidean geometry as the two-dimensional space for geographical theories.

A fractal is a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension (i.e., the dimension in Euclidean space, e.g., the topological dimension of a point is 0, of a line, 1, of a planar object, 2, and of a volume, 3). In contrast, the Hausdorff-Besicovitch dimension can be non-integer. It can be calculated with the box counting method, which may be applied to two-dimensional images as follows: For resolution $1/2^k$, where $N(1/2^k)$ denotes the minimum number (i.e., counting) of squares (i.e., boxes) of side-length 2^k needed to cover all of the lines forming a two-dimension image, the Hausdorff-Besicovitch dimension, *d*, is given by

$$d = \lim_{k \to \infty} \frac{\ln N(1/2^k)}{k \ln 2}$$

The dimension *d* is always greater than or equal to its topological counterpart: $d = 0.90309$ for the set of rational numbers (points with a topological dimension of 0), and $d = 1$ for a straight line (a line has a topological dimension of 1).

Manhattan (taxicab) geometry is non-Euclidean, and replaces the Pythagorean theorem with a metric that equals the sum of the absolute values of the differences between a pair of coordinates. The plane has a square grid superimposed on it, resulting in a regular square grid of points. The rotation, but not reflection or translation, transformation of this lattice grid affects Manhattan distances. This space may be defined as follows: A two-dimensional Manhattan space comprises P^2 points (u_i, v_i) forming a regular square *P*-by-*P* grid, together with links between pairs of nearby points that are exactly one unit of separation apart. The resulting plane geometry figure is a regular square tessellation, whereas the resulting graph is a regular square lattice. A family of specific finite Manhattan spaces is given by *P* ranging from 2 to infinity.

The purpose of this paper is to summarize results for analyses on a unit square upon which the density (i.e., the number of vertical and horizontal lines) of a lattice grid increases. Empirical examples motivate this exploration, which links Manhattan space to fractals as well as to spatial statistics. Fractal dimensions have been calculated with the Fractalyse software[2].

1.3. A summary of empirical evidence

Selected real-world transportation networks were analyzed (Griffith et al. 2010) to determine whether or not empirical evidence implies that physical space is more Manhattan than Euclidean in nature. First, minimum paths between selected points in the hierarchical set of expressways, highways, and streets for the Lansing, MI metropolitan region were analyzed. The Manhattan metric furnishes the function that best characterizes these minimum paths. Furthermore, the distance function for the restricted access expressways is closer to a Manhattan metric, whereas the distance function for the combination of all three types of roads is roughly midway between a Manhattan and a Euclidean metric.

Next, minimum paths between all stations in the following four limited-access urban mass-transit rail networks (Griffith et al. 2010) were analyzed: Pittsburgh, Dallas-Ft. Worth, Toronto, and Washington, DC. Again, the distance function for each of these sets of minimum paths lies between a Manhattan and a Euclidean metric. In these cases, the distance function for the near-linear Pittsburgh light rail network is closer to a Manhattan metric, whereas the distance function for the spider-like Washington, DC subway is closer to a Euclidean metric.

Finally, minimum paths between residential houses and arterial road exits in the Whitehills and Bailey limited-access neighborhoods (Griffith et al. 2010) located in the Lansing, MI metropolitan region were analyzed. Once more, the distance function for each neighborhood lies between that for a Manhattan and a Euclidean space. In both cases, it is much closer to Manhattan space.

The principal conclusions here are as follows:
1) distance functions fall between those for a Manhattan and a Euclidean space; and,
2) a positive relationship appears to exist between the Minkowskian distance function exponent and the fractal dimension of selected empirical network graphs.

These findings imply that theoretical work based upon fractals and the Manhattan distance metric should be illuminating.

1.4. Manhattan space: infill asymptotics

The simplest Manhattan space forms a square with four points. If these links and points coincide with the borders and corners of a unit square, Manhattan spaces can be constructed that involve

[2] For benchmark purposes, the fractal dimensions were calculated for the following images: a straight line, a Sierpiński carpet, a Koch coastline, a Koch island, a Sierpiński gasket, and a filled rectangle.

an increasing number of vertical and horizontal lines occurring between these borders. As the number of these lines goes to infinity, the Manhattan space lattice grid converges on a filled-in unit square.

1.4.1. Lattice-based fractal dimension results

The fractal dimension of a given *P*-by-*P* Manhattan space lattice has the following entries in its box counting regression, for which vector $\mathbf{N}(k) = [(n_1(k), n_2(k),..., n_K(k)]$

$$n_i(k) = \begin{cases} 2k\ln 2 & i = 1,2,\ldots,P-1, \\ \ln\left(P\left(2^{k+1} - P\right)\right) & i = P, P+1,\ldots K \end{cases}$$

where $\mathbf{f}(2)$ is some function of 2, and $K > P$ denotes the finest resolution used in the box counting procedure. *K* is often 14 in practice. If all entries in the vector $\mathbf{N}(k)$ are $ln[P(2^{k+1} - P)]$, then $\mathbf{f}(2)$ reduces to 2^k, and hence the fractal dimension is $k = 1$. Because the smallest values of *P* is 2, at least one entry in the vector $\mathbf{N}(k)$ is $2kln(2)$, and hence *k* must be greater than 1. In other words, the smallest fractal dimension for a Manhattan space is greater than 1. Meanwhile, as *P* goes to infinity, *k* increases. When *K* is infinity, all entries in the vector $\mathbf{N}(k)$ are $2kln(2)$, $\mathbf{f}(2)$ reduces to 2^k, and hence the fractal dimension is $k = 2$. Exploratory work reveals a numerical relationship between the two quantities of *P* and its corresponding fractal dimension, calculated with Fractalyse.

The principal implications here are:

1) as *P* increases, the bounded space becomes increasingly filled, converging on a bounded Euclidean plane; and,
2 the fractal dimension of finite Manhattan space relates to the principal eigenfunction of the corresponding regular lattice planar graph.

1.4.2. Approximate Euclidean distances in Manhattan space

The difference between Manhattan and Euclidean distance between a pair of points is given by

$$|u_2 - u_1| + |v_2 - v_1| - \sqrt{(u_2 - u_1)^2 + (v_2 - v_1)^2} = g + h - \sqrt{g^2 + h^2} = \delta$$

because all points are uniformly spaced along both their horizontal and their vertical axes. The relationships between *g* and *h* are as follows (see Fig. 1):

$g = h : 2(P - g)^3$

$g < h : 2(P - g)(P - h)(P - h + 1)$

The maximum acceptable difference quantity δ determines the degree of similarity between Manhattan and Euclidean distances (see Fig. 2). Because the largest distance between points in a unity square is √2, setting δ to this value results in a 100% agreement between distances rendered by the two metric functions. Setting δ to 0 results in a 100[2*P*/(*P*2 + 1)]% agreement, which asymptotically goes to 0%. Fig. 2b portrays cases between these two extreme values of δ. These trajectories indicate a decline for small values of *P*

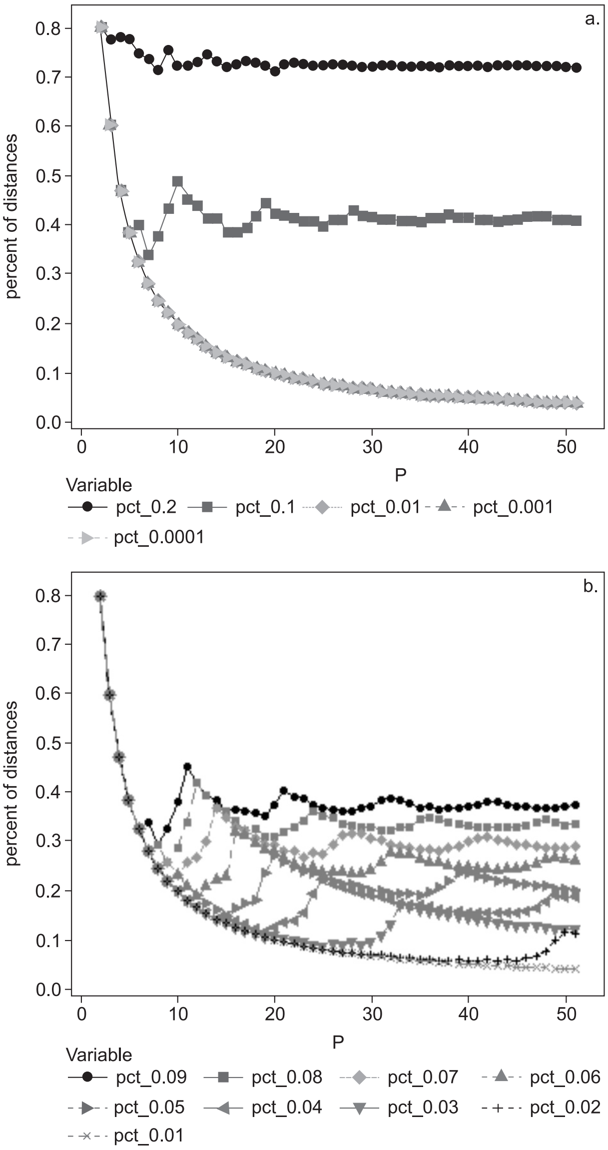

Fig. 2. Percentage of approximately equal Manhattan and Euclidean distances with increasing *P*. Left (a): for thresholds ranging from 0.00001 to 0.2. Right (b): for thresholds ranging from 0.01 to 0.09.

until Manhattan space contains a sufficient density of lines, at which time the trajectories begin to increase and oscillate as they converge on their asymptotic values. If δ = 0.2 (i.e., up to a 14.1% increase in Manhattan distance over Euclidean distance), for example, then the asymptotic percentage of acceptable differences is roughly 75. If δ = 0.01, this agreement decreases to 0.2%.

2. Prospect

These preceding, and other, experiments challenge us to pursue more deeply possible abstract linkages between theory and practice. Thus, we see a number of interesting research directions for consideration.

2.1. Pattern compression

Hyperbolic geometry is another non-Euclidean geometry in which parallel lines do meet, and in such a way that the points at infinity are included in the model. One way to visualize the hyperbolic plane is as the Euclidean plane with all horizon points, points at infinity, included. One model of this plane, the Poincaré disk, permits the entire hyperbolic plane to be compressed into a bounded space. In contrast, the Euclidean plane has infinite, unbounded extent, and cannot be so compressed. Visualize a stereographic projection of the sphere from the north pole into a plane tangent to the sphere at the south pole. The projection of the north pole forces the attempted compactification to infinity.

An opportunity for such compression might arise in new settings as well as in casting existing research into this non-Euclidean framework. The image in Fig. 3 comprises two copies (one reflected) of a fractal construction generated as a compression pattern for optimizing land/water interface in a marina (Arlinghaus & Nystuen 1990). When it is viewed in this manner, as two copies, we see it through the non-Euclidean lens of the Poincaré disk in which the parallel lines of the road edges do meet. (Maurits Escher's Circle Limit series suggests beautiful illustrations of this sort of idea.) Carried to the limit, this construction, given that the generator has 4 sides, and that the self-similarity factor is also 4, leads to a dimension of 2. That is, $F = [\log(n)/\log(k^{0.5})]$, where n is the number of generator sides and k is some factor of self-similarity.

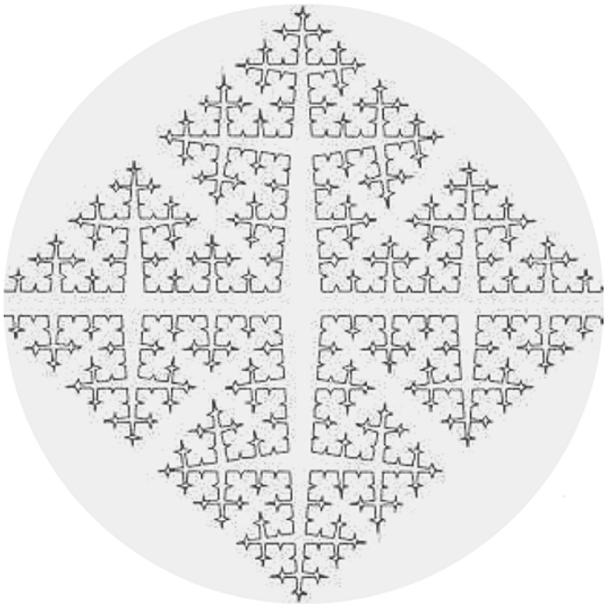

Fig. 3. The top half of this figure is reflected about a horizontal equator to create the bottom half. Imagine the whole figure embedded in a circumscribing disk.

Not only is there the sort of forward and back opportunity of existing and future research, there is also an opportunity to cross disciplinary boundaries. Network compression is important within various existing graph-theoretical or network science structures. One early example appears with Steiner networks — networks of shortest total length (Arlinghaus 1977 and later). Clear evidence of the importance of compression when dealing with numbers also exists. The Fibonacci sequence permits data compression as a universal code (Wikipedia 2010). Considering if, and how, the non-Euclidean world translates into the electronic environment, parallel processing, and various interesting coding situations based on compression patterns of various sorts, particularly as these relate to geographic information (Arlinghaus et al. 1989), merits attention. The realm of possibilities, at the theoretical level, is endless.

2.2. Turning theory into practice

Often theory, regardless of how abstract it may be, furnishes a basis for conceptualizing real-world example situations. Fractal geometry constitutes one of the more abstract theories that are capable of achieving this end.

2.2.1. Models as an intervening step

Consider a specific urban landscape with an underlying grid road pattern, as suggested in the previous fractal analysis of empirical grids, in order to facilitate ease and logic of movement. Consider endowing that pattern with a set of separated zoning categories reflecting human decisions of various sorts. As an intervening step in turning theory into practice in an actual city, consider the Burgess model or other artificial expressions of urban spatial structure and growth. The concentric circles of the Burgess model reflect different densities of land use, and therefore different densities of the underlying road pattern. Thus, application of a single fractal iteration sequence would not generate appropriate pattern, or its compression. Imagine, therefore, that in the core zone containing the downtown, a fractal iteration sequence of squares that, when carried to the limit has a dimension of 2, is truncated so that generated parcel size corresponds to observed parcel size (e.g., minimum, maximum, average, or as desired) according to zoning type. In the transition ring, choose an iteration pattern for roads that has a fractal dimension less than 2, and again truncate that sequence according to parcel size demands. In the small-house residential ring, choose a pattern that has a fractal dimension less than the previous two rings, and again truncate it according to parcel size. In an outer ring, representing large housing, choose a pattern with an even lower fractal dimension, and again truncate according to land-use parcel size.

What happens with pattern within these rings is straightforward: create a fractal pattern that generates a road grid and truncate this pattern generation in accordance with parcel size as determined by land use and zoning. What is more difficult is to consider how to align that pattern at the edges of adjacent rings. Typically, the adjacent patterns do not mesh because different iteration sequences, reflecting variety in space-filling based on land use, are employed. One approach might be to align core road network boundary points. The left-over points become cul-de-sacs, which might be used to advantage for privacy needs for residential land uses or for industrial uses wishing a relatively low profile. Here, unlike the case of much existing planning, good reason exists to create cul-de-sacs — as an appendix-style vestigial organ of an otherwise efficiently structured network.

Generally, a grid exists throughout a given landscape, but the space-filling needs of different zoning categories produce distinct neighborhoods of road network. Within-ring accessibility along a grid network is encouraged, while between-ring accessibility is reduced, perhaps fostering a safer environment. Such an approach also offers a visual "interest" factor in an otherwise boring overall grid pattern. Naturally, one might vary the urban model, the method for partitioning urban areas, as well as the general planning context (beyond zoning and land use). Transcending the models, which are useful as transitions from theory to practice, we see the importance of real-world implementation of ideas and ideals.

2.2.2. Ideas for real-world implementation

Models serve as "pilots" for ideas, as "beta-tests" for strategies. To move to the real-world from the conceptualized model-world is an art. No fixed way exists to do so. Often, temporarily discarding some of the structure from the model-world is helpful, especially when some element of it returns at a later stage in real-world suggestions for implementation.

Thus, when moving from the previously outlined Burgess idea, one might not wish to begin by first establishing abstract fractal pattern and then assessing how well actual road networks fit abstract form; rather, actual networks might serve as the starting point. A straightforward manner to calculate road density within a region is simply to measure the length of road network within the region and divide by the radius of the region (if it is circular), its perimeter, its area, or some other general regional measure. Such an index might be applied in concentric circles or other regions.

A density measure of this sort, unlike a fractal dimension, is a relative measure. Different answers might come about when they should not, simply by shifting a boundary a bit, or choosing a different radius. Relative measures offer opportunity for the "gerrymandering" of results. One way to overcome this relative measure problem might be to employ the fractal measures that depend on global whole-hierarchy characteristics rather than only on local ones. The existing fractal space-filling calculations (as previously discussed, or perhaps for hexagons; Arlinghaus 1985) may be used to calibrate the measure as a "fractally-bound density measure" for urban road network accessibility. Thus, in the inner zone, where highest space filling is required, the density measure would not be allowed to go beyond the highest fractal space-filling value. Subsequent circles or zones are bounded by subsequent fractal space-filling values, until the edge of the compact region is reached: all of which assumes that no issues exist involving points at infinity in relation to Euclidean geometry.

2.3. Turning practice back into theory

Often real-world examples suggest further needs for theory. Boundary value problems arise in the previously outlined Burgess idea. They occur because we split the domain of the function used for fractal calculation. Good real-world reasons may exist for splitting a domain. The reasons for doing so should be compelling (as with zoning), and might include a wide variety of considerations: rivers with few crossing points are one possibility; barriers, such as the Berlin Wall, might suggest others that limit access. Accessibility based on visible features might be one criterion for partitioning the domain of a function. Cataloguing/classifying the multitude of possible geographic reasons for domain partitioning is an interesting problem itself, particularly when considering associated boundary value issues, and assembling wisdom gained from specific styles of application. Taxonomy is the heart of science; it, too, has boundaries and exemplifies the boundary value problem. One future research direction concerning this topic might be to create a taxonomic document about boundary value problems in geography and the associated splitting of geographical domains.

From an abstract viewpoint, the splitting of domains is not desirable because calculation from the left of a boundary value may not produce the same results as calculation from the right. Thus, one prefers entire domain calculations rather than split domain calculations. As few boundary points as possible are best, although, as previously noted, they may be desirable from a real-world point of view. Reconciliation of conflicting views may come about in a variety of ways. One reason hyperbolic geometry is attractive for real-world application is that even the points at infinity are included: no boundary exists that separates them from the other points, and hence no boundary value problems are present!

We wish to proceed with this general research agenda, individually and collaboratively, in various ways. Hopefully, others will see much more, and advance our inventory and prospect of such analysis even more rapidly than the current rate of city growth.

3. Conclusions and implications

In conclusion, this paper contributes an interesting discussion to the urban growth and fractality literature, in both theoretical and practical terms. It furnishes a brief, selective inventory of related ideas from the past, and an overview of empirical findings concerning fractals and the non-Euclidean geometry of Manhattan space. The empirical geographic work involves metric function characterizations of minimum path distances between places in urban space, which are best described by a general Minkowskian one whose parameters are between those for Manhattan and Euclidean space. These Minkowskian metric results also relate to the fractal dimensions of the underlying physical spaces. One principal implication of this work is that theoretical, as well as applied, ideas based upon fractals and the Manhattan distance metric merit attention when formulating or re-formulating mathematical spatial theory.

We also outline some prospective work derived from implications of the empirical results. The Burgess (i.e., concentric zones), Hoyt (i.e., sector), and Harris-Ullman (i.e., multiple nuclei) models of urban spatial structure furnish good candidates for reformulations that incorporate fractal structures of city street networks. We outline one possibility involving the Burgess model. In summary, considerably more research effort needs to be devoted to urban growth and fractality.

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